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THE SECOND VARIATION FORMULA FOR WILLMORE SUBMANIFOLDS IN S^n

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The second variation formula for Willmore submanifolds in S^n *

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Abstract: In [8] the third author gave a submanifold theory of Moebius geometry in S^n and calculated the first variation formula of the Willmore functional by using Moebius invariants. In this paper we present the second variation formula for Willmore submanifolds. As an application of these variation formulas we give the standard examples of Willmore hypersurfaces $\{W_k^n := S^k(\sqrt{(n-k)/n}) \times S^{n-k}(\sqrt{k/n}),\ 1 \le k \le n-1\}$ in S^{n+1} (which can be obtained by exchanging radii in the Clifford tori $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$) and show that they are stable Willmore hypersurfaces. In case of surface in S^3, the stability of the Cliford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ is given by J. L. Weiner in [9].

Keywords: Willmore functional, The second variation formula, Stability, Willmore conjecture

1991 MR Subject Classification 53A30, 53B25

§0. Introduction

Let M be an m-dimensional submanifold immersed in S^n. The Willmore functional W is defined by

$$W(M) = \left(\frac{m}{m-1}\right)^{\frac{m}{2}} \int_M \left(S - m\|H\|^2\right)^{\frac{m}{2}} dM, \tag{0.1}$$

where S is the square of the length of the second fundamental form, H is the mean curvature vector and dM is the volume element of the induced metric on M. In case of surface in S^3, this functional is equivalent to the Willmore functional $W_e(M) := 4\int_M H^2 dM$ for surfaces in R^3, which is well-studied (cf. [2], [4], [7]). An equivalent version of Willmore conjecture states that $W(T^2) \ge 8\pi^2$ for any embedded trous T^2 in S^3 and the equality holds if and only if T^2 is Moebius equivalent to the Clifford torus. Many effords and partial results have been made for this conjecture, but it remains open since 1965.

From the analysis point of view the second variation formula of Willmore functional would be very important towards the conjecture. In 1978 J. L. Weiner gave the second

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is in particular a Dupin hypersurface. Thus we know from Thorbergsson Theorem ([T]) that the number γ of distinct principal curvatures for compact Möbius isoparametric hypersurfaces embedded in $\mathbf{S}^{n+1}$ can only take the values $\gamma = 2, 3, 4, 6$. In this paper we classify all Möbius isoparametric hypersurfaces with $\gamma = 2$. Our main result is the following

Classification Theorem. *Let $x : \mathbf{M} \to \mathbf{S}^{n+1}$ be a Möbius isoparametric hypersurface with two distinct principal curvatures. Then x is Möbius equivalent to an open part of one of the following hypersurfaces in $\mathbf{S}^{n+1}$:*

(i) *the torus $\mathbf{S}^k(a) \times \mathbf{S}^{n-k}(b)$ with $1 \le k \le n-1$ and $a^2 + b^2 = 1$;*

(ii) *the pre-image of a stereographic projection of the standard cylinder $\mathbf{S}^k(1) \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1}$ with $1 \le k \le n-1$;*

(iii) *the pre-image of a stereographic projection of the cone in $\mathbf{R}^{n+1}$:*

$$\tilde{x}(u, v, w) = \left(\frac{r}{\sqrt{1+r^2}} vu, \frac{1}{\sqrt{1+r^2}} v, w\right), \quad (u, v, w) \in \mathbf{S}^k(1) \times \mathbf{R}^+ \times \mathbf{R}^{n-k-1}$$

with $1 \le k \le n-2$.

Since a Dupin hypersurface in $\mathbf{S}^{n+1}$ with two distinct principal curvature is a Möbius isoparametric hypersurface (cf. Corollary 3.6), this theorem can be regarded as the local version of the classification theorem of Cecil and Ryan (cf. [C-R-1], Theorem 6.2) of complete Dupin hypersurfaces with two distinct principal curvatures embedded in $\mathbf{R}^{n+1}$. In our local case the hypersurfaces (iii) appear. The method we use here is the framework of Möbius differential geometry given by the third author in [W], which is completely different to that of Cecil and Ryan. We should mention also that by a theorem of Pinkall in [P] all hypersurfaces in (i), (ii) and (iii) are equivalent under Lie sphere transformation group.

This paper is organized as following. In §2 we give Möbius invariants and structure equations we need. In §3 we study the three examples of Möbius isoparametric hypersurfaces (i), (ii) and (iii) in the classification theorem. Then we prove the classification theorem in §4.

§2. Möbius invariants for hypersurfaces in $\mathbf{S}^{n+1}$.

In this section, we give Möbius invariants and structure equations for hypersurfaces in $\mathbf{S}^{n+1}$. For more detail we refer to [W].

Let $x : \mathbf{M} \to \mathbf{S}^{n+1}$ be a hypersurface without umbilic point. Let $I = dx \cdot dx$ and II be the first and second fundamental forms, H the mean curvature of x. We define the Möbius position vector $Y : \mathbf{M} \to \mathbf{R}^{n+3}_1$ of x by

$$Y = \rho(1, x), \quad \rho = \sqrt{\frac{n}{n-1}} ||II - HI||. \tag{2.1}$$

Then we have

Theorem 2.1.([W]) *Let x, $\tilde{x} : \mathbf{M} \to \mathbf{S}^{n+1}$ be two submanifolds without umbilic point. Then x and $\tilde{x}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+2, 1)$ in $\mathbf{R}^{n+3}_1$ such that $Y = \tilde{Y} T$.*

It follows immediately from Theorem 2.1 that

$$g = < dY, dY > = \rho^2 dx \cdot dx \tag{2.2}$$

is a Möbius invariant metric on $\mathbf{M}$, where $< \ , \ >$ is the inner product in $\mathbf{R}_1^{n+3}$ defined by

$$< X, W > = -x_0 y_0 + x_1 y_1 + \cdots + x_{n+2} y_{n+2}, \tag{2.3}$$

where $X = (x_0, x_1, \cdots, x_{n+2})$ and $W = (y_0, y_1, \cdots, y_{n+2})$.

Let Δ be the Laplacian operator with respect to g. We define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2} < \Delta Y, \Delta Y > Y, \tag{2.4}$$

then we have

$$< Y, Y > = < N, N > = 0, \quad < Y, N > = 1. \tag{2.5}$$

Let $\{E_1, E_2, \cdots, E_n\}$ be a local orthonormal basis for $(\mathbf{M}, g)$ with dual basis $\{\omega_1, \omega_2, \cdots, \omega_n\}$. We write $Y_i = E_i(Y)$. Then by (2.2) and (2.4) we have

$$< Y_i, Y > = 0, \ < Y_i, N > = 0, \ < Y_i, Y_j > = \delta_{ij}. \tag{2.6}$$

Since $\{Y, N, Y_1, \cdots, Y_n\}$ is a subbasis for $\mathbf{R}_1^{n+3}$, we can find a smooth map $\xi : \mathbf{M} \to \mathbf{R}_1^{n+3}$ such that

$$< \xi, \xi > = 1, \ < \xi, Y > = < \xi, N > = < \xi, Y_i > = 0, \ 1 \le i \le n. \tag{2.7}$$

We call $\xi : \mathbf{M} \to \mathbf{S}_1^{n+2}$ the Möbius Gauss map of x. Using the frame $\{Y, N, Y_1, \cdots, Y_n, \xi\}$ in $\mathbf{R}_1^{n+3}$ along $\mathbf{M}$ and the formulas (2.5), (2.6) and (2.7) we can write the structure equations for Y by

$$E_i(N) = \sum_j A_{ij} Y_j + C_i \xi \tag{2.8}$$

$$E_j(Y_i) = -A_{ij} Y - \delta_{ij} N + \sum_k \Gamma_{ij}^k Y_k + B_{ij} \xi \tag{2.9}$$

$$E_i(\xi) = -C_i Y - \sum_j B_{ij} Y_j, \tag{2.10}$$

where $\{\Gamma_{ij}^k\}$ is the Levi-Civita connection for the Möbius metric g, and

$$\mathbb{A} = \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \quad \mathbb{B} = \sum_{ij} B_{ij} \omega_i \otimes \omega_j, \quad \Phi = \sum_i C_i \omega_i \tag{2.11}$$

are Möbius invariants. We call $\mathbb{B}$ the Möbius second fundamental form and Φ Möbius form. The relations between Möbius invariants and Euclidean invariants of x are given by (cf. [W])

$$\begin{aligned} A_{ij} = & - \rho^{-2}(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho) e_j(\log\rho) - H h_{ij}) \\ & - \frac{1}{2}\rho^{-2}(||\nabla \log\rho||^2 - 1 + H^2)\delta_{ij}; \end{aligned} \tag{2.12}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}); \tag{2.13}$$

$$C_i = - \rho^{-2}(e_i(H) + \sum_j (h_{ij} - H\delta_{ij}) e_j(\log\rho)); \tag{2.14}$$

where $\{e_i\}$ is a local orthonormal basis for $I = dx \cdot dx$, (Hess_{ij}) and ∇ are Hessian-matrix and gradient with respect to I.

Let $\{R_{ijkl}\}$ be the curvature tensor of g and $\mathcal{K}$ its normalized scalar curvature. Then we can write the integrability conditions for the structure equations (2.8)-(2.10) by

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k \tag{2.15}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{kj}A_{ki}) \tag{2.16}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j \tag{2.17}$$

$$R_{ijkl} = (B_{ik}B_{jl} - B_{il}B_{jk}) + (\delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}) \tag{2.18}$$

$$\sum_i A_{ii} = \frac{1}{2n}(1 + n^2\mathcal{K}) \tag{2.19}$$

$$\sum_i B_{ii} = 0, \quad \sum_{ij} B_{ij}^2 = \frac{n-1}{n}, \tag{2.20}$$

where $\{A_{ij,k}\}$, $\{B_{ij,k}\}$ and $\{C_{i,j}\}$ are covariant derivatives of $\mathbb{A}$, $\mathbb{B}$ and Φ with respect to the Möbius metric g.

Let S be the Weingarten operator for x. We call

$$\mathbb{S} = \rho^{-1}(S - H\mathrm{id}) \tag{2.21}$$

the Möbius shape operator of x. Since for $n \geq 3$ all coeficients in (2.8)-(2.10) are determined by $(g, \mathbb{S})$, we have

Theorem 2.2.([W]) *Two hypersurfaces x, $\tilde{x} : \mathbf{M} \to \mathbf{S}^{n+1}$ $(n \geq 3)$ without umbilic point are Möbius equivalent if and only if there exists a diffemorphism $\sigma : \mathbf{M} \to \mathbf{M}$ which preserves the Möbius metric and the Möbius shape operator.*

§3. Möbius isoparametric hypersurfaces in $\mathbf{S}^{n+1}$.

We recall that a hypersurface $x : \mathbf{M} \to \mathbf{S}^{n+1}$ without umbilic point is called Möbius isoparametric, if its Möbius form Φ defined by (1.1) vanishes and its Möbius shape operator $\mathbb{S} = \rho^{-1}(S - H\mathrm{id})$ has contant eigenvalues.

Let $x : \mathbf{M} \to \mathbf{S}^{n+1}$ be a hypersurface without umbilic point. Let $\{\kappa_1, \cdots, \kappa_\gamma\}$ be the distinct principal curvatures of S. We have the follwoing decomposition of $\mathbf{TM}$ into eigenspaces of S:

$$\mathbf{TM} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots \oplus \mathbb{V}_\gamma, \quad \gamma \geq 2, \tag{3.1}$$

where $\mathbb{V}_i$ is the eigenspace corresponding to κ_i.

Definition 3.1. *$x : \mathbf{M} \to \mathbf{S}^{n+1}$ is called a Dupin hypersurface if for any $1 \leq i \leq \gamma$ we have (i) $\dim \mathbb{V}_i$ = constant and (ii) $X(\kappa_i) = 0$ for all $X \in \mathbb{V}_i$.*

Since the Möbius shape operator $\mathbb{S} = \rho^{-1}(S - H\mathrm{id})$ has the same eigenspaces as S, we know that $\mathbb{V}_i$ is also the eigenspace of $\mathbb{S}$ corresponding to the eigenvalues $\rho^{-1}(\kappa_i - H)$. Thus the number of distinct principal curvatures of $\mathbb{S}$ is also γ.

Proposition 3.2. *Any Möbius isoparametric hypersurface $x : \mathbb{M} \to \mathbb{S}^{n+1}$ is in particular a Dupin hypersurface .*

Proof. For any nonzero vector $X \in \mathbb{V}_i$ we define $e_i = \frac{X}{||X||}$. We extend e_i to a basis $\{e_1, e_2, \cdots, e_i, \cdots, e_n\}$ for $\mathbf{TM}$ consisting of eigenvectors of S. Since Φ in (1.1) vanishes, we get

$$C_i = -\rho^{-2}\{e_i(H) + (\kappa_i - H)e_i(\log \rho)\} = -\rho^{-2}\{e_i(\kappa_i) - \rho e_i(\rho^{-1}(\kappa_i - H))\} = 0, \tag{3.2}$$

Since the eigenvalues $\rho^{-1}(\kappa_i - H)$ of $\mathbb{S}$ is constant, we get $e_i(\kappa_i) = 0$, i.e., $X(\kappa_i) = 0$. □

It follows immediately from (3.2) that

Corollary 3.3. *Let $x : \mathbb{M} \to \mathbb{S}^{n+1}$ be a hypersurface whose Möbius shape operator has constant eigenvalues. Then x is a Dupin hypersurface if and only if it is a Möbius isoparametric hypersurface.*

It is proved by Thorbergsson in [T] that the number γ of distinct principal curvatures of a compact Dupin hypersurface embedded in $\mathbb{S}^{n+1}$ can only take value $\gamma = 1, 2, 3, 4, 6$. As a consequence of Proposition 3.2 we get

Theorem 3.4. *The number γ of distinct principal curvatures of a compact Möbius isoparametric hypersurface embedded in $\mathbb{S}^{n+1}$ can only take the values $\gamma = 2, 3, 4, 6$.*

Proposition 3.5. *Let $x : \mathbb{M} \to \mathbb{S}^{n+1}$ be a hypersurface with two distinct principal curvatures λ and μ $(\lambda > \mu)$ of multiplicity k and $n-k$ respectively, then the eigenvalues of its Möbius shape operator $\mathbb{S}$ are constant, which are given by*

$$\kappa_1 = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}, \quad \kappa_2 = -\frac{1}{n}\sqrt{\frac{(n-1)k}{n-k}}. \tag{3.3}$$

Proof. We have

$$H = \frac{1}{n}(k\lambda + (n-k)\mu), \quad ||II||^2 = k\lambda^2 + (n-k)\mu^2, \tag{3.4}$$

which implies that

$$\rho^2 = \frac{n}{n-1}(||II||^2 - nH^2) = \frac{k(n-k)}{n-1}(\lambda - \mu)^2. \tag{3.5}$$

Thus the eigenvalues κ_1 and κ_2 of $\mathbb{S}$ are given by

$$\kappa_1 = \rho^{-1}(\lambda - H) = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}, \quad \kappa_2 = \rho^{-1}(\mu - H) = -\frac{1}{n}\sqrt{\frac{(n-1)k}{n-k}}. \tag{3.6}$$

□

It follows from Proposition 3.5 and Corollary 3.3 that

Corollary 3.6. *Let $x : \mathbf{M} \to \mathbf{S}^{n+1}$ be a Dupin hypersurface with two distinct principal curvatures. Then x is a Möbius isoparametric hypersurface.*

Standard examples of Möbius isoparametric hypersurfaces are the images of (Euclidean) isoparametric hypersurfaces in $\mathbf{S}^{n+1}$ under Möbius transformations, because for these hypersurfaces the Möbius form Φ in (1.1) vanishes and the eigenvalues of $\mathbb{S}$ are constant. In the rest of this section we give examples of Möbius isoparametric hypersurfaces in $\mathbf{S}^{n+1}$ with $\gamma = 2$.

Our first example is the isoparametric torus $x : \mathbf{S}^k(a) \times \mathbf{S}^{n-k}(b) \to \mathbf{S}^{n+1}$ defined by

$$x = (ax_1, bx_2), \quad a^2 + b^2 = 1, \quad a > 0, \quad b > 0, \tag{3.7}$$

where $x_1 : \mathbf{S}^k \to \mathbf{R}^{k+1}$ and $x_2 : \mathbf{S}^{n-k} \to \mathbf{R}^{n-k+1}$ are unit spheres. The Weingarten operator S of x has two distinct eigenvalues $\lambda = \frac{b}{a}$ and $\mu = -\frac{a}{b}$ with multiplicity k and $n-k$ respectively. Using (3.5) we get $\rho^2 = \frac{k(n-k)}{n-1}a^{-2}b^{-2}$. Thus the Möbius metric g of x is given by

$$g = \rho^2 \mathrm{d}x \cdot \mathrm{d}x = \frac{k(n-k)}{(n-1)b^2}\mathrm{d}x_1 \cdot \mathrm{d}x_1 + \frac{k(n-k)}{(n-1)a^2}\mathrm{d}x_2 \cdot \mathrm{d}x_2 := g_1 + g_2. \tag{3.8}$$

If we take a local orthonormal basis $\{e_1, \cdots, e_k\}$ for $(\mathbf{S}^k, g_1)$ and a local orthonormal basis $\{e_{k+1}, \cdots, e_n\}$ for $(\mathbf{S}^{n-k}, g_2)$, then under the basis $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ the Möbius shape operator $\mathbb{S}$ is given by the diagonal matrix

$$\mathbb{S} = \mathrm{diag}(\overbrace{\kappa_1, \cdots, \kappa_1}^{k}, \overbrace{\kappa_2, \cdots, \kappa_2}^{n-k}), \tag{3.9}$$

where κ_1 and κ_2 are given by (3.3). It is clear that $x : \mathbf{S}^k(a) \times \mathbf{S}^{n-k}(b) \to \mathbf{S}^{n+1}$ is a Möbius isoparametric hypersurface with $\gamma = 2$.

The second example is the pre-image of a stereographic projection σ of the standard cylinder $\tilde{x} : \mathbf{S}^k(1) \times \mathbf{R}^{n-k} \to \mathbf{R}^{n+1}$. The hypersurface $x = \sigma^{-1} \circ \tilde{x} : \mathbf{S}^k(1) \times \mathbf{R}^{n-k} \to \mathbf{S}^{n+1}$ has two distinct principal curvatures. If we take a local basis $\{e_1, \cdots, e_k\}$ for $\mathbf{S}^k(1)$ and $\{e_{k+1}, \cdots, e_n\}$ for $\mathbf{R}^{n-k}$, then under the basis $\{e_1, \cdots, e_k, e_{k+1}, \cdots, e_n\}$ the Möbius shape operator $\mathbb{S}$ is given by (3.9). Since

$$g = \rho^2 \mathrm{d}x \cdot \mathrm{d}x = \tilde{\rho}^2 \mathrm{d}\tilde{x} \cdot \mathrm{d}\tilde{x}, \tag{3.10}$$

where $\tilde{\rho}^2 = \frac{n}{n-1}(||\tilde{II}||^2 - n\tilde{H}^2)$, $\tilde{II}$ is the second fundamental form of $\tilde{x} : \mathbf{S}^k(1) \times \mathbf{R}^{n-k} \to \mathbf{R}^{n+1}$, $\tilde{H}$ its mean curvature (cf. [W], (4.33) and (4.35)), we know that the Weingarten operator of $\tilde{x}$ has two eigenvalues 1 and 0 of multiplity k and $n-k$ respectively. So we have $\tilde{\rho}^2 = \frac{k(n-k)}{n-1}$. Thus we get

$$g = \frac{k(n-k)}{n-1}\mathrm{d}u \cdot \mathrm{d}u + \frac{k(n-k)}{n-1}\mathrm{d}v \cdot \mathrm{d}v, \ (u, v) \in \mathbf{S}^k(1) \times \mathbf{R}^{n-k}. \tag{3.11}$$

Since $\tilde{x}$ is clearly a Dupin hypersurface with $\gamma = 2$, then $x = \sigma^{-1} \circ x$ is also a Dupin hypersurface with $\gamma = 2$. Thus by Corollary 3.6 we know that x is a Möbius isoparametric hypersurface in $\mathbf{S}^{n+1}$.

Finally we consider the cone $\tilde{x}: \mathbf{S}^k \times \mathbf{R}^+ \times \mathbf{R}^{n-k-1} \to \mathbf{R}^{n+1}$:

$$\tilde{x}(u,v,w) := (\frac{r}{\sqrt{1+r^2}}vu, \frac{1}{\sqrt{1+r^2}}v, w), \ (u,v,w) \in \mathbf{S}^k \times \mathbf{R}^+ \times \mathbf{R}^{n-k-1}, r \in \mathbf{R}. \tag{3.12}$$

Let $\sigma : \mathbf{S}^{n+1} \to \mathbf{R}^{n+1} \cup \{\infty\}$ be a stereographic projection. We define $x = \sigma^{-1} \circ \tilde{x} : \mathbf{S}^k \times \mathbf{R}^+ \times \mathbf{R}^{n-k-1} \to \mathbf{S}^{n+1}$. By (3.12) we have

$$d\tilde{x} = (\frac{r}{\sqrt{1+r^2}}(dvu + vdu), \frac{1}{\sqrt{1+r^2}}dv, dw). \tag{3.13}$$

We know that

$$\tilde{n} = \frac{1}{\sqrt{1+r^2}}(-u, r, 0) \tag{3.14}$$

is the unit normal of $\tilde{x}$. Thus we have

$$d\tilde{x} \cdot d\tilde{x} = \frac{r^2}{1+r^2}v^2 du \cdot du + dv^2 + dw \cdot dw, \tag{3.15}$$

$$\tilde{II} = -d\tilde{x} \cdot d\tilde{n} = \frac{r}{1+r^2}vdu \cdot du, \tag{3.16}$$

which implies that $\tilde{x}$ has two distinct principal curvatures $\frac{1}{rv}$ and 0 of multiplitity k and $(n-k)$ respectively at the point $(u,v,w) \in \mathbf{S}^k \times \mathbf{R}^+ \times \mathbf{R}^{n-k-1}$. Thus $x = \sigma^{-1} \circ \tilde{x}$ has two distinct principal curvatures of multiplicity k and$(n-k)$ respectively. For any basis $\{e_1, \cdots, e_k\}$ for $\mathbf{S}^k$ and $\{e_{k+1}, \cdots, e_n\}$ for $\mathbf{R}^+ \times \mathbf{R}^{n-k-1}$ the Möbius shape operator is given by (3.9). Since

$$\tilde{\rho}^2 = \frac{n}{n-1}(\frac{k}{r^2v^2} - \frac{k^2}{nr^2v^2}) = \frac{k(n-k)}{(n-1)r^2v^2}, \tag{3.17}$$

we get

$$g = \rho^2 dx \cdot dx = \tilde{\rho}^2 d\tilde{x} \cdot d\tilde{x} = \frac{k(n-k)}{(n-1)r^2}du \cdot du + \frac{k(n-k)}{(n-1)r^2v^2}(dv^2 + dw \cdot dw). \tag{3.18}$$

Since $\tilde{x}$ is clearly a Dupin hypersurface in $\mathbf{R}^{n+1}$ with $\gamma = 2$, then $x = \sigma^{-1} \circ \tilde{x}$ is a Dupin hypersurface in $\mathbf{S}^{n+1}$ with $\gamma = 2$. By Corollary 3.6 we know that x is a Möbius isoparametric hypersurface with $\gamma = 2$.

§4. The proof of the classification theorem.

In this section we prove the classification theorem in §1. Since in case $n = 2$ the theorem have be proved in [W], we may assume that $n \geq 3$, thus we can apply Theorem 2.2.

Let $x : \mathbf{M} \to \mathbf{S}^{n+1}$ be a Möbius isoparametric hypersurface with two distinct principal curvatures of multiplicity k and $(n-k)$. Without loss of generality we may assume that $\mathbf{M}$ is simply connected. Then we have the following decomposition:

$$\mathbf{TM} = \mathbb{V}_1 \otimes \mathbb{V}_2,$$

where $\mathbb{V}_1$ and $\mathbb{V}_2$ are the eigenspaces of $\mathbb{S}$ with eigenvalues

$$\kappa_1 = \frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}, \ \kappa_2 = -\frac{1}{n}\sqrt{\frac{(n-1)k}{n-k}}$$

respectively. We make the following convention on the ranges of indices:

$$1 \le i,j,l,m \le k, \quad k+1 \le \alpha,\beta,\gamma,\lambda \le n, \quad 1 \le a,b,c \le n. \tag{4.1}$$

Let $\{E_i\}$ be a local orthonormal basis for $\mathbb{V}_1$ and $\{E_\alpha\}$ be a local orthonormal basis for $\mathbb{V}_2$ with respect to the Möbius metric g. Then under the basis $\{E_i, E_\alpha\}$ for **TM** we have

$$B_{ij} = \kappa_1\delta_{ij}, \quad B_{\alpha\beta} = \kappa_2\delta_{\alpha\beta}, \quad B_{i\alpha} = 0. \tag{4.2}$$

Since $C_a \equiv 0$, we know from (2.15) and (2.17) that $\{B_{ab,c}\}$ and $\{A_{ab,c}\}$ are totally symmetric. It follows from (2.16) and (4.2) that

$$A_{i\alpha} = 0, \quad 1 \le i \le k, \quad k+1 \le \alpha \le n. \tag{4.3}$$

From the formula

$$dB_{ij} + \sum_l B_{il}\omega_{lj} + \sum_\alpha B_{i\alpha}\omega_{\alpha j} + \sum_l B_{lj}\omega_{li} + \sum_\alpha B_{\alpha j}\omega_{\alpha i} = \sum_a B_{ij,a}\omega_a \tag{4.4}$$

and (4.2) we get $B_{ij,a} = 0$.
Similarly we get from

$$dB_{\alpha\beta} + \sum_l B_{\alpha l}\omega_{l\beta} + \sum_\gamma B_{\alpha\gamma}\omega_{\gamma\beta} + \sum_l B_{l\beta}\omega_{l\alpha} + \sum_\gamma B_{\gamma\beta}\omega_{\gamma\alpha} = \sum_a B_{\alpha\beta,a}\omega_a \tag{4.5}$$

and (4.2) that $B_{\alpha\beta,a} = 0$. It follows that

$$B_{ab,c} \equiv 0. \tag{4.6}$$

Thus we have

$$dB_{i\alpha} + \sum_l B_{l\alpha}\omega_{li} + \sum_\beta B_{\beta\alpha}\omega_{\beta i} + \sum_l B_{il}\omega_{l\alpha} + \sum_\beta B_{\alpha\beta}\omega_{\beta i} = \sum_a B_{i\alpha,a}\omega_a = 0. \tag{4.7}$$

Using (4.2) and (4.7) we get

$$\omega_{i\alpha} = 0, \quad 1 \le i \le k, \quad k+1 \le \alpha \le n, \tag{4.8}$$

which implies that

$$R_{i\alpha ab} = 0. \tag{4.9}$$

Using (4.2) and (2.18) we obtain

$$0 = R_{i\alpha j\beta} = B_{ij}B_{\alpha\beta} + \delta_{ij}A_{\alpha\beta} + \delta_{\alpha\beta}A_{ij}. \tag{4.10}$$

By letting (i) $i \neq j, \alpha = \beta$; (ii) $i = j, \alpha \neq \beta$; (iii) $i = j, \alpha = \beta$ we get

$$A_{ij} = 0, \ \ i \neq j, \quad A_{\alpha\beta} = 0, \ \ \alpha \neq \beta, \quad A_{i\alpha} = 0; \tag{4.11}$$

$$\kappa_1\kappa_2 + A_{\alpha\alpha} + A_{ii} = 0, \ \ 1 \leq i \leq k, \ \ k+1 \leq \alpha \leq n. \tag{4.12}$$

It follows from (4.12) that

$$A_{11} = A_{22} = \cdots = A_{kk} = \sigma, \quad A_{(k+1)(k+1)} = \cdots = A_{nn} = \tau. \tag{4.13}$$

Using the formula

$$\mathrm{d}A_{i\alpha} + \sum_l A_{l\alpha}\omega_{li} + \sum_\beta A_{\beta\alpha}\omega_{\beta i} + \sum_j A_{ij}\omega_{j\alpha} + \sum_\beta A_{i\beta}\omega_{\beta\alpha} = \sum_a A_{i\alpha,a}\omega_a \tag{4.14}$$

and (4.8), (4.11) we get $A_{i\alpha,a} = 0$, i.e.,

$$A_{i\alpha,j} = 0, \quad A_{i\alpha,\beta} = 0, \quad 1 \leq i,j \leq k, \quad k+1 \leq \alpha,\beta \leq n. \tag{4.15}$$

From the formula

$$\mathrm{d}A_{ij} + \sum_l A_{lj}\omega_{li} + \sum_\alpha A_{\alpha j}\omega_{\alpha i} + \sum_l A_{il}\omega_{lj} + \sum_\alpha A_{i\alpha}\omega_{\alpha j} = \sum_a A_{ij,a}\omega_a \tag{4.16}$$

and (4.15), (4.11) we get

$$E_\alpha(A_{ij}) = 0, \quad 1 \leq i,j \leq k, \quad k+1 \leq \alpha \leq n. \tag{4.17}$$

Similarly from the formula

$$\mathrm{d}A_{\alpha\beta} + \sum_l A_{l\beta}\omega_{l\alpha} + \sum_\gamma A_{\gamma\beta}\omega_{\gamma\alpha} + \sum_l A_{\alpha l}\omega_{l\beta} + \sum_\gamma A_{\alpha\gamma}\omega_{\gamma\beta} = \sum_a A_{\alpha\beta,a}\omega_a \tag{4.18}$$

and (4.15), (4.11) we get

$$E_i(A_{\alpha\beta}) = 0, \quad 1 \leq i \leq k, \quad k+1 \leq \alpha,\beta \leq n. \tag{4.19}$$

Thus (4.2), (4.12), (4.17) and (4.19) yield that the functions σ and τ defined by (4.13) are constant. Since $\omega_{i\alpha} = 0$, we have

$$\mathrm{d}\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \mathrm{d}\omega_\alpha = \sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta. \tag{4.20}$$

Thus the eigenspaces $\mathbb{V}_1$ and $\mathbb{V}_2$ of the Möbius shape operator $\mathbb{S}$ are integrable. We can write $\mathbf{M} = \mathbf{M}_1 \times \mathbf{M}_2$ for some simply connected manifolds $\mathbf{M}_1$ of dimension k and $\mathbf{M}_2$ of dimension $n-k$. Moreover, if we define $g_1 = \sum_j \omega_j^2$ and $g_2 = \sum_\alpha \omega_\alpha^2$, then we have

$$(\mathbf{M}, g) = (\mathbf{M}_1, g_1) \times (\mathbf{M}_2, g_2). \tag{4.21}$$

It follows from (2.18), (4.2) and (4.11) that

$$R_{ijkl} = (\kappa_1^2 + 2\sigma)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}); \tag{4.22}$$

$$R_{\alpha\beta\gamma\lambda} = (\kappa_2^2 + 2\tau)(\delta_{\alpha\gamma}\delta_{\beta\lambda} - \delta_{\alpha\lambda}\delta_{\beta\gamma}). \tag{4.23}$$

Thus $(\mathbf{M}_1, g_1)$ and $(\mathbf{M}_2, g_2)$ are spaces of constant curvature. By (3.3) we have

$$\kappa_1\kappa_2 = -\frac{n-1}{n^2} < 0,$$

then we know from (4.12) that at least one of $\{\sigma, \tau\}$ must be positive. Without loss of generality we assume that $\sigma > 0$. We consider the following three cases: (i) $\kappa_2^2 + 2\tau > 0$; (ii) $\kappa_2^2 + 2\tau = 0$ and (iii) $\kappa_2^2 + 2\tau < 0$.

In the case $\kappa_2^2 + 2\tau > 0$ we define

$$a = \sqrt{\frac{k(n-k)(\kappa_2^2 + 2\tau)}{n-1}}, \quad b = \sqrt{\frac{k(n-k)(\kappa_1^2 + 2\sigma)}{n-1}}. \tag{4.24}$$

Then by (3.3) and (4.12) we get $a^2 + b^2 = 1$. Let $\tilde{x} : \mathbf{S}^k(a) \times \mathbf{S}^{n-k}(b) \to \mathbf{S}^{n+1}$ be hypersurface defined by (3.7). Then by (3.8) and (4.24) the Möbius metric $\tilde{g}$ of $\tilde{x}$ is given by $\tilde{g} = \tilde{g_1} + \tilde{g_2}$, where

$$\tilde{g_1} = \frac{1}{\kappa_1^2 + 2\sigma} dx_1 \cdot dx_1, \quad \tilde{g_2} = \frac{1}{\kappa_2^2 + 2\tau} dx_2 \cdot dx_2. \tag{4.25}$$

Since by (4.22), (4.23) and (4.25) both $(\mathbf{S}^k(a), \tilde{g_1})$ and $(\mathbf{M}_1, g_1)$, $(\mathbf{S}^{n-k}(b), \tilde{g_2})$ and $(\mathbf{M}_2, g_2)$ are constant curvature spaces with the same curvature, we can find isometries

$$\phi_1 : (\mathbf{M}_1, g_1) \to (\mathbf{S}^k(a), \tilde{g_1}), \ \phi_2 : (\mathbf{M}_2, g_2) \to (\mathbf{S}^{n-k}(b), \tilde{g_2}). \tag{4.26}$$

Thus $\phi = (\phi_1, \phi_2) : \mathbf{M} \to \mathbf{S}^k(a) \times \mathbf{S}^{n-k}(b)$ preserves the Möbius metric and the Möbius operator (cf. Proposition 3.5 and (3.9)). From Theorem 2.2 we know that $x : \mathbf{M} \to \mathbf{S}^{n+1}$ is Möbius equivalent to $\tilde{x} : \mathbf{S}^k(a) \times \mathbf{S}^{n-k}(b) \to \mathbf{S}^{n+1}$ as the first example in §3.

In the case $\kappa_2^2 + 2\tau = 0$ we get from (4.12) and (3.3) that

$$\kappa_1^2 + 2\sigma = \kappa_1^2 + \kappa_2^2 + 2(\sigma + \tau) = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 = \frac{n-1}{k(n-k)}. \tag{4.27}$$

Using the same argument as in the first case and (3.11) one can show that $x : \mathbf{M} \to \mathbf{S}^{n+1}$ is Möbius equivalent to the pre-image of a stereographic projection of the cylinder $\mathbf{S}^k(1) \times \mathbf{R}^{n-k}$.

Finally we consider the case $\kappa_2^2 + 2\tau < 0$. Since by (4.12) and (3.3) we have

$$\kappa_1^2 + 2\sigma > (\kappa_1^2 + 2\sigma) + (\kappa_2^2 + 2\tau) = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 = \frac{n-1}{k(n-k)}, \tag{4.28}$$

we can find a unique $r > 0$ satisfying

$$\kappa_1^2 + 2\sigma = \frac{n-1}{k(n-k)}(1+r^2). \tag{4.29}$$

It follows from (4.28) that

$$\kappa_2^2 + 2\tau = -\frac{n-1}{k(n-k)}r^2. \tag{4.30}$$

Using the same argument as in the first case and (3.18) one can easily show that $x : \mathbf{M} \to \mathbf{S}^{n+1}$ is Möbius equivalent to the pre-image of a stereographic projection of the cone given by (3.12).

This completes the proof of the classification theorem.

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